Chapter 3
Theory of Equations

In this chapter we learn some theory about equations, particularly about polynomial equations, and their solutions; we study some properties of polynomial equations, formation of polynomial equations with given roots, the fundamental theorem of algebra, and to know about the number of positive and negative roots of a polynomial equation.

Using these ideas we reach our goal of solving polynomial equations of certain types. We also learn to solve some non–polynomial equations using techniques developed for polynomial equations

LEARNING OBJECTIVES

Upon completion of this chapter, the students will be able to

- form polynomial equations satisfying given conditions on roots
- demonstrate the techniques to solve polynomial equations of higher degree
- solve equations of higher degree when some roots are known to be complex or surd, irrational and rational
- find solutions to some non-polynomial equations using techniques developed for polynomial equations
- identify and solve reciprocal equations
- determine the number of positive and negative roots of a polynomial equation using Descartes Rule

3.2 Basics of Polynomial Equations

3.2.1 Different types of Polynomial Equations

We already know that, for any non–negative integer , a polynomial of degree n in one variable *x* is an expression given by

$$\boxed{P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0} \qquad \ldots \quad (3.1)$$

with $a_n \neq 0$ and $a_r \in \mathbb{C}$, for all $r = 0, 1, 2, \ldots, n$.

When all the coefficients of a polynomial P are real, we say

"*P* is a polynomial over $\mathbb{R}$".

Similarly we use terminologies like

"*P* is a polynomial over $\mathbb{C}$",

"*P* is a polynomial over $\mathbb{Q}$", and is a polynomial over $\mathbb{Z}$".

The function *P* defined by

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad \text{... (3.1)}$$

is called a polynomial function. The equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \qquad \text{... (3.2)}$$

is called a polynomial equation.

If $a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a_0 = 0$

for some $c \in \mathbb{C}$, then *c* is called a zero of the polynomial (3.1) and root or solution of the polynomial equation (3.2).

We recall that $\sin^2 x + \cos^2 x = 1$ is an identity on $\mathbb{R}$, while $\sin x + \cos x = 1$ and $\sin^3 x + \cos^3 x = 1$ are equations.

3.2.2 Quadratic Equations

For the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, $b^2 - 4ac$ is called the discriminant and it is usually denoted by Δ. We know that $\dfrac{-b+\sqrt{\Delta}}{2a}$ and $\dfrac{-b-\sqrt{\Delta}}{2a}$ are roots of the quadratic equation $ax^2 + bx + c = 0$. The two roots together are usually written as $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

We also learnt that $\Delta = 0$ if, and only if, the roots are equal. When a, b, c are real, we know

- $\Delta > 0$ if, and only if, the roots are real and distinct
- $\Delta < 0$ if, and only if, the quadratic equation has no real roots.

3.3 Vieta's Formulae and Formation of Polynomial Equations

3.3.1 Vieta's formula for Quadratic Equations

Let α and β be the roots of the quadratic equation $ax^2+bx+c=0, a\neq 0$. Then

$$ax^2+bx+c = a(x-\alpha)(x-\beta) = ax^2 - a(\alpha+\beta)x + a(\alpha\beta) = 0$$

Equating the coefficients of like powers, we see that

$$\alpha+\beta = \frac{-b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

So **a** quadratic equation whose roots are α and β is $x^2-(\alpha+\beta)x+\alpha\beta=0$; that is, **a** quadratic equation with given roots is,

$$x^2 - (\text{sum of the roots})\ x + \text{ product of the roots} = 0.$$

Theorem 3.3.1 **(The Fundamental Theorem of Algebra)**

Every polynomial equation of degree $n \geq 1$ has at least one root in $\mathbb{C}$.

Using this, we can prove that a polynomial equation of degree n has at least n roots in $\mathbb{C}$ when the roots are counted with their multiplicities. This statement together with our discussion above says that

a polynomial equation of degree n has exactly n roots in $\mathbb{C}$ when the roots are counted with their multiplicities.

Some authors state this statement as the fundamental theorem of algebra.

Vieta's Formula for Polynomial equation of degree 2

Vieta's Formula for Polynomial equation of degree 2

Let us start with a quadratic equation

$$ax^2 + bx + c = 0, a \neq 0.$$

By the fundamental theorem of algebra, it has two roots.

Let α and β be the roots. Then we have the Vieta's formula as

$$\alpha + \beta = \frac{-b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

Vieta's Formula for Polynomial equation of degree 3

Now we obtain these types of relations to higher degree polynomials. Let us consider a general cubic equation

$$ax^3 + bx^2 + cx + d = 0, a \neq 0.$$

By the fundamental theorem of algebra, it has three roots. Let α, β, and γ be the roots. Thus we have

$$ax^3 + bx^2 + cx + d = a(x-\alpha)(x-\beta)(x-\gamma)$$

Expanding the right hand side, gives

$$ax^3 - a(\alpha+\beta+\gamma)x^2 + a(\alpha\beta+\beta\gamma+\gamma\alpha)x - a(\alpha\beta\gamma).$$

Comparing the coefficients of like powers, leads to

$$\alpha+\beta+\gamma = \frac{-b}{a}, \quad \alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a} \text{ and } \alpha\beta\gamma = \frac{-d}{a}.$$

Since the degree of the polynomial equation is 3, we must have $a \neq 0$ and hence division by a is meaningful. If a monic cubic polynomial has roots α, β, and γ, then

$$\text{coefficient of } x^2 = -(\alpha+\beta+\gamma),$$

$$\text{coefficient of } x = \alpha\beta+\beta\gamma+\gamma\alpha, \text{ and}$$

$$\text{constant term} = -\alpha\beta\gamma.$$

Vieta's Formula for Polynomial equation of degree $n>3$

The same is true for higher degree monic polynomial equations as well. If a monic polynomial equation of degree n has roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, then

$$
\begin{array}{lcl}
\text{coefficient of } x^{n-1} & = & -\sum \alpha_1 \\
\text{coefficient of } x^{n-2} & = & \sum \alpha_1 \alpha_2 \\
\text{coefficient of } x^{n-3} & = & -\sum \alpha_1 \alpha_2 \alpha_3 \\
\vdots & \vdots & \vdots \\
\text{coefficient of } x & = & (-1)^{n-1} \sum \alpha_1 \alpha_2 \ldots \alpha_{n-1} \\
\text{constant term} & = & (-1)^{n} \alpha_1 \alpha_2 \ldots \alpha_n
\end{array}
$$

Formation of Polynomial Equations with given Roots

A cubic polynomial equation whose roots are α , β , and γ is

$$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma=0\,.$$

A polynomial equation of degree n with roots $\alpha_1,\alpha_2,\ldots,\alpha_n$ is given by

$$x^n-\left(\sum\alpha_1\right)x^{n-1}+\left(\sum\alpha_1\alpha_2\right)x^{n-2}-\left(\sum\alpha_1\alpha_2\alpha_3\right)x^{n-3}+\cdots+(-1)^n\,\alpha_1\alpha_2\cdots\alpha_n=0$$

Theorem 3.4.1 (Complex Conjugate Root Theorem)

If a complex number z_0 is a root of a polynomial equation with real coefficients, then its complex conjugate $\bar{z}_0$ is also a root

As every integer is a rational number, we know that every real number is also a complex number. So to clearly specify a complex number that is not a real number, that is to specify members of form $\alpha + i\beta$ with $\beta \neq 0$, we use the term **"non-real complex number"**. Some authors call such a number an **imaginary number**.

Let $z_0 = \alpha + i\beta$ with $\beta \neq 0$. Then $\bar{z}_0 = \alpha - i\beta$. If $\alpha + i\beta$ is a root of a polynomial equation $P(x) = 0$ with real coefficients, then by Complex Conjugate Root Theorem, $\alpha - i\beta$ is also a root of $P(x) = 0$.

Usually the above statement will be stated as ***complex roots occur in pairs*****;** but actually it means that ***non-real complex roots or imaginary roots occur as conjugate pairs*****,** ***if the coefficients of the polynomial equation are real*****.**

If p is real, discuss the nature of the roots of the equation $4x^2+4px+p+2=0$, in terms of p .

Solution. The discriminant $\Delta=(4p)^2-4(4)(p+2)=16\left(p^2-p-2\right)=16(p+1)(p-2)$. So

$$\Delta<0 \text{ if } -1<p<2$$

$$\Delta=0 \text{ if } p=-1 \text{ or } p=2$$

$$\Delta>0 \text{ if } -\infty<p<-1 \text{ or } 2<p<\infty$$

Thus the given polynomial has

imaginary roots if $-1<p<2$;

equal real roots if $p=-1$ or $p=2$;

distinct real roots if $-\infty<p<-1$ or $2<p<\infty$.

Theorem

Let p and q be rational numbers such that $\sqrt{q}$ is irrational.

If $p+\sqrt{q}$ is a root of a quadratic equation with rational coefficients,

then $p-\sqrt{q}$ is also a root of the same equation.

"For a polynomial equation with rational coefficients, irrational roots occur in pairs" is not true.

Theorem

Let p and q be rational numbers so that $\sqrt{p}$ and $\sqrt{q}$ are irrational numbers; further let one of $\sqrt{p}$ and $\sqrt{q}$ be not a rational multiple of the other. If $\sqrt{p}+\sqrt{q}$ is a root of a polynomial equation with rational coefficients, then $\sqrt{p}-\sqrt{q}, -\sqrt{p}+\sqrt{q}$, and $-\sqrt{p}-\sqrt{q}$ are also roots of the same polynomial equation.

3.6 Roots of Higher Degree Polynomial Equations

General Solutions of a Cubic Equation

General solutions of the cubic equation $ax^3 + bx^2 + cx + d = 0$ are given by

$$P + Q - \frac{b}{3a},\ \omega P + \omega^2 Q - \frac{b}{3a} \quad \text{and} \quad \omega^2 P + \omega Q - \frac{b}{3a}$$

Where ω is a non real cube root of 1 and

$$P = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} \qquad Q = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}$$

$$p = \frac{c}{a} - \frac{b^2}{3a^2} \quad \text{and} \quad q = \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}$$

General Solutions of a Quartic Equation

General solutions of the quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ are given by

$$-\frac{b}{4a} \pm S \pm \frac{1}{2}\sqrt{-4S^2 - 2p + \frac{q}{S}}$$

where

$$S = \frac{1}{2}\sqrt{-\frac{2}{3}p + \frac{1}{3a}\left(Q + \frac{\Delta_0}{Q}\right)} \quad Q = \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}$$

$$\Delta_0 = c^2 - 3bd + 12ae \quad \Delta_1 = 2c^3 - 9bcd + 27b^2e + 27ad^2 - 72ace$$

$$p = \frac{8ac - 3b^2}{8a^2} \text{ and } q = \frac{b^3 - 4abc + 8a^2d}{8a^3}$$

General Solutions of Higher Degree Equations

A solution of a polynomial equation written only using its coefficients, the four basic arithmetic operators (addition, multiplication, subtraction and division), and rational exponentiation (power to a rational number, such as square, cube, square roots, cube roots and so on) is called a radical solution. Abel proved that it is impossible to write a radical solution for general polynomial equation of degree five or more.

3.6.1 Some Results on Polynomial Equations

We state a few results about polynomial equations that are useful in solving higher degree polynomial equations.

- Every polynomial equation in one variable is a continuous function from $\mathbb{R}$ to $\mathbb{R}$.
- For a polynomial equation $P(x)=0$ of even degree, $P(x)$ tends to ∞ as x tends to $-\infty$ and as x tends to ∞. Thus the graph of an even degree polynomial equation seems to start from left top and ends at right top.
- All results discussed on "graphing functions" in Volume I of eleventh standard textbook can be applied to the graphs of polynomials. For instance, a change in the constant term of a polynomial moves its graph up or down only.
- Every polynomial equation is differentiable any number of times.
- The real roots of a polynomial equation $P(x)=0$ are the points on the x-axis where the graph of $P(x)=0$ cuts the x-axis.
- If a and b are two real numbers such that $P(a)$ and $P(b)$ are of opposite signs, then
  - there is a point c on the real line for which $P(c)=0$.
  - that is, there is a root between a and b.
  - it is not necessary that there is only one root between such points; there may be $3,5,7,\ldots$ roots; that is the number of real roots between a and b is odd and not even.

3.7 Polynomials with Additional Information

3.7.1 Imaginary Roots

If is an imaginary root of a quartic polynomial with real coefficients, then is also a root

Surds Roots

If is a root of a quadric polynomial equation with rational coefficients, then is also a root;

3.7.2 Polynomial equations with Even Powers Only

If $P(x)$ is a polynomial equation of degree $2n$, having only even powers of x, then by replacing x^2 by y, we get a polynomial equation with degree n in y; let $y_1, y_2, \cdots y_n$ be the roots of t his polynomial equation. Then considering the n equations $x^2 = y_r$, we can find two values for x each r; these $2n$ numbers are the roots of the given polynomial equation in x.

3.7.3 Zero Sum of all Coefficients

Let $P(x) = 0$ be a polynomial equation such that the sum of the coefficients is zero.

The sum of coefficients is nothing but $P(1)$. The sum of all coefficients is zero means that $P(1) = 0$

which says that 1 is a root of $P(x)$. The rest of the problem of solving the equation is easy.

3.7.4 Equal Sums of Coefficients of Odd and Even Powers

The sum of all coefficients of $P(x) = 0$ is zero and hence -1 is a root of $P(-x) = 0$

which says that 1 is a root of $P(-x) = 0$.

The sum of all coefficients of $P(-x) = 0$ is zero and hence 1 is a root of $P(-x) = 0$

which says that -1 is a root of $P(x) = 0$.

3.7.5 Roots in Progressions

3.7.6 Partly Factored Polynomials

3.8 Reciprocal Equations

Definition 3.8.1:

A polynomial P of degree n is said to be a reciprocal polynomial if

$$P(x) = x^n P\left(\frac{1}{x}\right) \text{ or } P(x) = -x^n P\left(\frac{1}{x}\right).$$

The equation $P(x) = 0$ is called a reciprocal equation.

Theorem 3.8.2

If $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0 \ (a_n \neq 0)$,

is a reciprocal polynomial, then one and only one of the following two statements is true:

(i) $a_{n-r} = a_r$ for all $r = 0, 1, 2, \ldots, n$.

(ii) $a_{n-r} = -a_r$ for all $r = 0, 1, 2, \ldots, n$.

The condition "$a_{n-r} = a_r$ for all r" says that

$$a_n = a_0, a_{n-1} = a_1, a_{n-2} = a_2, \cdots$$

The condition "$a_{n-r} = -a_r$ for all r" says that

$$a_n = -a_0, a_{n-1} = -a_1, a_{n-2} = -a_2, \cdots$$

3.8.3 Non-polynomial Equations

Some non-polynomial equations can be solved using polynomial equations. As an example let us consider the equation $\sqrt{15-2x} = x$. First we note that this is not a polynomial equation. Squaring both sides, we get $x^2 + 2x - 15 = 0$. We know how to solve this polynomial equation. From the solutions of the polynomial equation, we can analyse the given equation. Clearly 3 and -5 are solutions of $x^2 + 2x - 15 = 0$. If we adopt the notion of assigning only nonnegative values for $\sqrt{\bullet}$ then $x = 3$ is the only solution; if we do not adopt the notion, then we get $x = -5$ is also a solution

3.9 Descartes Rule

In this section we discuss some bounds for the number of positive roots, number of negative roots and number of nonreal complex roots for a polynomial over . These bounds can be computed using a powerful tool called “Descartes Rule”.

3.9.1 Statement of Descartes Rule

To discuss the rule we first introduce the concept of change of sign in the coefficients of a polynomial.

Consider the polynomial.

$$2x^7 - 3x^6 - 4x^5 + 5x^4 + 6x^3 - 7x + 8$$

For this polynomial, let us denote the sign of the coefficients using the symbols ‘ $+$ ’ and ‘ $-$ ’as $+,-,-,+,+,-,+$

Note that we have not put any symbol corresponding to x^2.

We further note that 4 changes of sign occurred (at x^6, x^4, x^1 and x^0).

Definition 3.9.1

A ***change of sign in the coefficients is said to occur*** *at the* j^{th} *power of* x *in a polynomial* $P(x)$, *if the coefficient of* x^{j+1} *and the coefficient of* x^{j} *are of different signs. (For zero coefficient we take the sign of the immediately preceding nonzero coefficient.)*

From the number of sign changes, we get some information about the roots of the polynomial using ***Descartes Rule****. As the proof is beyond the scope of the book, we state the theorem without proof.*

Theorem 3.9.1 (Descartes Rule)

If p is the number of positive zeros of a polynomial $P(x)$ with real coefficients and s is the number of sign changes in coefficients of $P(x)$, then $s-p$ is a nonnegative even integer.

The theorem states that the number of positive roots of a polynomial $P(x)$ cannot be more than the number of sign changes in coefficients of $P(x)$. Further it says that the difference between the number of sign changes in coefficients of $P(x)$ and the number of positive roots of the polynomial $P(x)$ is even.

As a negative zero of $P(x)$ is a positive zero of $P(-x)$ we may use the theorem and conclude that the number of negative zeros of the polynomial $P(x)$ cannot be more than the number of sign changes in coefficients of $P(-x)$ and the difference between the number of sign changes in coefficients of $P(-x)$ and the number of negative zeros of the polynomial $P(x)$ is even.

3.9.2 Attainment of bounds

The Descartes rule gives only upper bounds for the number of positive roots and number of negative roots; the Descartes rule neither gives the exact number of positive roots nor the exact number of negative roots. But we can find the exact number of positive, negative and nonreal roots in certain cases. Also, it does not give any method to find the roots.

3.9.3 Bounds for the number of Imaginary (Nonreal Complex)roots

Using the Descartes rule, we can compute a lower bound for the number of imaginary roots. Let m denote the number of sign changes in coefficients of $P(x)$ of degree n; let k denote the number of sign changes in coefficients of $P(-x)$. Then there are at least $n-(m+k)$ imaginary roots for the polynomial $P(x)$. Using the other conclusion of the rule, namely, the difference between the number of roots and the corresponding sign changes is even, we can sharpen the bounds in particular cases.

Thank you